

Exact differential and quantum corrections of entropy for axially symmetric black holes

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ABSTRACT: Using the exactness criteria of entropy from the first law of black hole thermodynamics, we study quantum corrections for axially symmetric black holes.

Hawking's work on black hole radiation and evaporation proved extremely significant for studying black holes as thermodynamical systems. Here we study quantum mechanical phenomenon in the context of classical theory of general relativity. We are interested in studying the changes in classical entropy of black hole due to these quantum effects. For a black hole of mass, M , angular momentum, J , and charge, Q , the first law of thermodynamics is $dM = TdS + \Omega dJ + \Phi dQ$, where T is the temperature, S entropy, Ω angular velocity and Φ electrostatic potential. We can also write this as

$$dS(M, J, Q) = \frac{1}{T}dM - \frac{\Omega}{T}dJ - \frac{\Phi}{T}dQ. \quad (1)$$

Now, we note that this differential in three parameters is exact if the following conditions are satisfied [1]

$$\frac{\partial}{\partial J} \left(\frac{1}{T} \right) = \frac{\partial}{\partial M} \left(-\frac{\Omega}{T} \right), \quad (2)$$

$$\frac{\partial}{\partial Q} \left(\frac{1}{T} \right) = \frac{\partial}{\partial M} \left(-\frac{\Phi}{T} \right), \quad (3)$$

$$\frac{\partial}{\partial Q} \left(-\frac{\Omega}{T} \right) = \frac{\partial}{\partial J} \left(-\frac{\Phi}{T} \right). \quad (4)$$

Thus entropy $S(M, J, Q)$ can be written in the integral form. Using this we work out quantum corrections of entropy [2, 3] beyond the semiclassical limit. Here we will apply this to axially symmetric static spacetimes.

We first consider the Kerr-Newman spacetime in Boyer-Lindquist coordinates

$$ds^2 = -\frac{\Delta^2}{\rho^2}(dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2}(adt - (r^2 + a^2)d\phi)^2,$$

where $\Delta^2 = (r^2 + a^2) - 2Mr + Q^2$, $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $a = \frac{J}{M}$.

The horizons for this metric are $r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}$. The outer horizon at r_+ is specified as the black hole horizon and is a null stationary 2-surface. The Killing vector normal to this surface is $\chi^\alpha = t^\alpha + \Omega \phi^\alpha$ and it is null on the horizon. This horizon is generated by the Killing vector χ^α , and the surface gravity κ associated with this Killing horizon is $\kappa^2 = \frac{-1}{2} \chi^{\alpha;\beta} \chi_{\alpha;\beta}$. Using this it is easy to evaluate the temperature $T = \kappa/2$ associated with this horizon as [1]

$$T = \left(\frac{\hbar}{2\pi} \right) \frac{\sqrt{M^4 - J^2 - Q^2 M^2}}{M \left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2} \right)}. \quad (5)$$

The angular velocity is $\Omega = J/M \left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2} \right)$, and the electrostatic potential becomes

$$\Phi = \frac{Q \left(M^2 + \sqrt{M^4 - J^2 - Q^2 M^2} \right)}{M \left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2} \right)}. \quad (6)$$

It is easy to see that these quantities for the Kerr-Newman black hole satisfy conditions (2)-(4), and therefore, dS is an exact differential. We use the modified surface gravity [4] due to quantum effects $\mathcal{K} = \mathcal{K}_0 \left(1 + \sum_i \frac{\alpha_i \hbar^i}{(r_+^2 + a^2)^i} \right)^{-1}$, where α_i correspond to higher order loop corrections to the surface gravity of black holes and $\mathcal{K} = 2\pi T$. Thus the entropy including the correction terms becomes

$$S = \frac{\pi}{\hbar} (r_+^2 + a^2) + \pi \alpha_1 \ln(r_+^2 + a^2) + \sum_{k \geq 2} \frac{\pi \alpha_{k-1} \hbar^{k-2}}{(2-k)(r_+^2 + a^2)^{k-2}} + \dots \quad (7)$$

Note that we can get the corrections for the Kerr black black hole [2] if we put charge $Q = 0$, the Schwarzschild black hole, $a = Q = 0$ and the Reissner-Nordström black hole, $a = 0$.

Using the Bekenstein-Hawking area law relating entropy and horizon area, $S = A/4\hbar$, where the area in our case is $A = 4\pi(r_+^2 + a^2)$, from (7) we obtain the modified area law as

$$S = \frac{A}{4\hbar} + \pi \alpha_1 \ln A - \frac{4\pi^2 \alpha_2 \hbar}{A} - \frac{8\pi^3 \alpha_3 \hbar^2}{A^2} - \dots \quad (8)$$

Now, we consider the stationary axisymmetric Einstein-Maxwell black holes in the presence of dilaton-axion field, found in heterotic string theory [5]. In Boyer-Lindquist coordinates these are described by [6]

$$ds^2 = -\frac{\Sigma - a^2 \sin^2 \theta}{\Delta} dt^2 - \frac{2a \sin^2 \theta}{\Delta} [(r^2 - 2Dr + a^2) - \Sigma] dt d\phi \\ + \frac{\Delta}{\Sigma} dr^2 + \Delta d\theta^2 + \frac{\sin^2 \theta}{\Delta} [(r^2 - 2Dr + a^2)^2 - \Sigma a^2 \sin^2 \theta] d\phi^2, \quad (9)$$

where $\Delta = r^2 - 2Dr + a^2 \cos^2 \theta$, $\Sigma = r^2 - 2Mr + a^2$.

They have the electric charge $Q = \sqrt{2\omega D(D-M)}$, where $\omega = e^d$. Here D , and d denote the dilaton charge and the massless dilaton field, and $m = M - D$ is the

Arnowitt-Deser-Misner (ADM) mass of the black hole. The electrostatic potential is $\Phi = (-2DM/Q(r_+^2 - 2Dr_+ + a^2))$. The angular velocity on the horizon is given by

$$\Omega = \frac{J}{2M \left[M(M + D) + \sqrt{M^2(M + D)^2 - J^2} \right]} \quad (10)$$

The metric has singularities at $r^2 - 2Dr + a^2 \cos^2 \theta = 0$. The outer and inner horizons are respectively $r_{\pm} = (M - Q^2/2\omega M) \pm \sqrt{(M - Q^2/2\omega M)^2 - a^2}$.

The Hawking temperature is [1]

$$T = \frac{\hbar}{4\pi} \left[\frac{\sqrt{M^2(M + D)^2 - J^2}}{M[M(M + D) + \sqrt{M^2(M + D)^2 - J^2}]} \right]. \quad (11)$$

One can easily check that the above thermodynamical quantities satisfy conditions (2)-(4). Thus the entropy differential dS is exact and we can work out the entropy corrections as

$$S = \frac{\pi}{\hbar} (r_+^2 - 2Dr_+ + a^2) + \pi\beta_1 \ln(r_+^2 - 2Dr_+ + a^2) + \sum_{k>2} \frac{\pi\beta_{k-1} \hbar^{k-2}}{(2-k)(r_+^2 - 2Dr_+ + a^2)^{k-2}} + \dots$$

The Bekenstein-Hawking entropy associated with this horizon is one quarter of the area of the horizon surface. It is important to note that unlike spherical geometry the horizon surface here is not simply a 2-sphere. The area of the horizon from the 2-metric on the horizon is $A = 4\pi(r_+^2 - 2Dr_+ + a^2)$, and using the corresponding entropy $S = \frac{A}{4\hbar}$, we again obtain the area law given by Eq. (8).

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